

# FLATS IN THE SPACE OF KÄHLER METRICS AND OKOUNKOV BODIES.

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ABSTRACT. We study sufficient conditions for the existence of flat subspaces in the space of continuous plurisubharmonic metrics on a polarised complex projective manifold, relying on the generalised Legendre transform to the Okounkov body defined by Witt Nyström, and a result of Schwer–Lytchak.

## INTRODUCTION.

In complex geometry, there is a well-known one-to-one correspondence between polarised toric varieties  $(X, L)$  and integral polytopes. Here,  $X$  is a complex projective manifold of dimension  $n$  with a dense action of  $\mathbb{G}_m^n$ , and  $L$  is an ample equivariant line bundle on  $X$ . One direction of this correspondence has been generalised by Kaveh–Khovanskii [10] and Lazarsfeld–Mustatá [12], building on an idea of Okounkov [16]: given *any* compact complex projective manifold  $X$  endowed with an ample line bundle  $L$ , the data of a complete flag of  $X$  allows one to construct a convex body  $\Delta(X, L)$  which captures important information about the asymptotics of the section ring of  $L$ . In the case where  $X$ ,  $L$ , and the flag are toric, one recovers the aforementioned polytope. In general, one can not "go back" in the construction of an Okounkov body: there is no guarantee that the construction is injective, and by [11] there are only countably many convex bodies arising as Okounkov bodies.

The toric correspondence can be refined on the level of *metrised* bundles and polytopes: given toric  $(X, L)$  and the associated polytope  $\Delta(X, L)$ , there is a bijection between toric continuous plurisubharmonic metrics on  $L$ , and convex functions on  $\Delta(X, L)$ . One obtains the convex function by taking the Legendre transform of the moment map associated to the metric on  $L$ , which is supported on the polytope  $\Delta$  by the fundamental work of Atiyah–Guillemin–Sternberg [1, 8]. Building on this correspondence and work of Boucksom–Chen [3], Witt Nyström [15] constructs a generalised Lebesgue transform (which he calls the *Chebyshev transform*) sending a continuous psh metric on an arbitrary ample line bundle  $L$  to a convex function on its Okounkov body.

In the toric case, the Legendre transform is also a geodesic-preserving map: plurisubharmonic geodesics in the sense of Mabuchi [14, 18] are mapped via the Legendre transform to affine segments in the space  $\text{Conv}(\Delta(X, L))$ . In light of the previously mentioned constructions, it is a very natural question to ask whether in the non-toric case, Mabuchi geodesics are also mapped to affine segments in the space of convex functions on the Okounkov body. The author gave a proof of this result in [17], which was pointed out to be wrong by Jakob Hultgren and Laszlo Lempert. The purpose of this note is to advertise an observation that any geodesically convex subset of the space of continuous psh metrics on  $L$  for which

this holds, is in fact *flat*, as we explain in more detail below. Corollarily, the main result of [17] cannot hold in general on the whole space of continuous psh metrics by [4].

In a negatively curved metric space  $(M, d)$ , a fundamental question is to understand when one can find or construct *flat subspaces*, i.e. isometric embeddings from a normed vector space (or a convex subset thereof) into  $M$ . Flat subspaces are very difficult to find in general: the most basic example is the embedding of a segment inside a one-dimensional vector space, which corresponds to finding geodesic segments between points in  $M$ . Darvas [7] introduced various metric structures  $d_p$ ,  $p \geq 1$  on the space of continuous psh metrics on an ample line bundle, which have some similarity with  $L^p$  structures on finite-dimensional vector spaces. Those structures have negative curvature in the sense that they satisfy a property called *Busemann convexity*, i.e. the distance function between two  $d_p$ -geodesics is a convex function of time. In the case  $p = 2$ , this even endows the space  $\text{CPSH}(X, L) := C^0 \cap \text{PSH}(X, L)$  with a CAT(0) structure. Our result therefore gives a sufficient condition to find (possibly infinite-dimensional) flats:

**Theorem A.** Let  $\mathcal{U}$  be a  $d_p$ -geodesically convex subset of  $\text{CPSH}(X, L)$ , for any (equivalently, all)  $p > 1$ . If there exists an admissible flag  $Y_\bullet$  of  $X$  such that, for any geodesic segment  $t \mapsto \varphi_t$  lying inside  $\mathcal{U}$ , the Chebyshev transform  $t \mapsto c_{Y_\bullet}[\varphi_t]$  is affine in  $\text{Conv}(\Delta_{Y_\bullet}(X, L))$ , then for all  $p > 1$ ,

$$(\mathcal{U}, d_p)$$

is isometric to a convex subset of a vector space endowed with a strictly convex norm.

We also give a similar result in the case  $p = 1$ , which requires more care (as Mabuchi geodesics are only a subset of all  $d_1$ -geodesics, much as is the case for the  $L^1$  distance in a finite-dimensional vector space) in Theorem 2.5.

We note that we also recover flatness of the space of invariant metrics in the toric case from Theorem A.

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## 1. PRELIMINARIES.

**1.1. Curvature in the space of metrics and  $d_p$ -distances.** Throughout,  $X$  will denote a complex compact projective manifold, and  $L$  will be an ample line bundle on  $X$ . We recall that a *plurisubharmonic metric* on  $L$  is a possibly singular, locally integrable metric  $\varphi$  on  $L$  such that  $dd^c\varphi \geq 0$  in the sense of currents. We denote the space of psh metrics on  $L$  by  $\text{PSH}(X, L)$ , and the space of continuous psh metrics by  $\text{CPSH}(X, L)$ .

Darvas [7] introduced, for  $p \geq 1$ , distances  $d_p$  on the space  $\text{CPSH}(X, L)$  induced by Finsler metrics, by setting for  $\varphi_0, \varphi_1$  smooth psh

$$d_p(\varphi_0, \varphi_1)^p := \inf_{\{\varphi_t\}_t} \int_0^1 |\dot{\varphi}_t|^p dt,$$

where the infimum is taken over smooth curves joining  $\varphi_0$  and  $\varphi_1$ , and the tangent space of the space of smooth psh metrics is identified with  $C^\infty(X)$ . (The  $d_p$ -metric completion of the space of smooth psh metrics is then shown to contain  $\text{CPSH}(X, L)$ .) For  $p > 1$ , the  $d_p$  metric makes CPSH into a uniquely geodesic space, and all  $d_p$  structures with  $p > 1$  share the same geodesics, which we call *Mabuchi geodesics*, a name historically justified by the case  $p = 2$  which was originally studied by Mabuchi [14]. In the case  $p = 1$ , Mabuchi geodesics are also  $d_1$ -geodesics, but those are not unique, much as is the case for the  $L^1$  metric on e.g. a finite-dimensional vector space. Existence of Mabuchi geodesics was shown by Chen [6], and various regularity results have been achieved for those [13]. Importantly, while  $\text{CPSH}(X, L)$  also admits a convex structure, and therefore "natural" geodesic segments given by straight line segments, Mabuchi geodesics do not coincide but in very specific cases with those straight line segments.

Indeed, the  $d_p$ -structures make  $\text{CPSH}(X, L)$  into a *negatively-curved* metric space, for various meanings of "negative curvature". By Chen–Cheng [5, Theorem 1.5] (see also Berman–Darvas–Lu [2, Proposition 5.1] for  $p = 1$ ), for all  $p \geq 1$ ,  $(\text{CPSH}(X, L), d_p)$  is *Busemann-convex*, i.e. given two geodesics  $t \mapsto \varphi_t, \psi_t$ , the function

$$t \mapsto d_p(\varphi_t, \psi_t)$$

is convex. This property is to be understood as a manifestation of negative curvature. In a somewhat different direction, by Calabi–Chen [4], the  $d_2$ -structure of Mabuchi makes CPSH into a negatively curved space in the sense of Alexandrov – i.e. a CAT(0) space. Flat subspaces in a CAT(0) space are to be understood as parts of the space with no curvature.

**1.2. Okounkov bodies and the generalised Legendre transform.** Let

$$Y_\bullet := Y_0 = X \supset Y_1 \cdots \supset Y_n = \{pt\}$$

be a complete flag of smooth irreducible subvarieties in  $X$ . One can define inductively a graded valuation

$$\nu_{Y_\bullet} : \bigoplus_k H^0(X, kL) \rightarrow \mathbb{Z}^{n+1}$$

by setting, for  $s \in H^0(X, kL)$ ,

$$\nu_{Y_\bullet}(s) := (k, \text{ord}_{Y_1}(s), \text{ord}_{Y_2}((s - \text{ord}_{Y_1}(s)Y_1)|_{Y_1}), \dots).$$

For all  $k$ , we denote by  $\Delta_{Y_\bullet}^k(X, L) \subset \{k\} \times \mathbb{Z}^n$  the image of  $H^0(X, kL)$  by  $\nu_{Y_\bullet}$ , and define the *Okounkov body*  $\Delta_{Y_\bullet}(X, L)$  to be the convex hull of the union of the projections of all  $k^{-1}\Delta_{Y_\bullet}^k(X, L)$  onto the last  $n$  variables. In particular, one has  $(L^d) = \text{vol}(\Delta_{Y_\bullet}(X, L))$  [12].

Building on Boucksom–Chen [3], Witt Nyström [15] defines a map  $c_{Y_\bullet} : \text{CPSH}(X, L) \rightarrow \text{Conv}(\Delta_{Y_\bullet}(X, L))$  as follows. The valuation  $\nu_{Y_\bullet}$  has *one-dimensional leaves* in the

sense of [10], i.e. it induces a graduation

$$H^0(X, kL) = \bigoplus_{\alpha \in \Delta_{Y_\bullet}^k(X, L)} gr_{Y_\bullet}^{k, \alpha}(X, L)$$

in complex lines. In local coordinates  $\underline{z} = (z_1, \dots, z_n)$  determined by the flag  $Y_\bullet$  about the point  $Y_n$ , a section  $s \in gr_{Y_\bullet}^{k, \alpha}(X, L)$  can be written as

$$s = c_\alpha \cdot z^{k\alpha} + \text{higher order terms.}$$

Let us denote by  $[z^{k\alpha}]$  the equivalence class of *monic* sections in that trivialisation, i.e. of the form

$$s = z^{k\alpha} + \text{higher order terms.}$$

One then defines

$$c_k[\varphi](k, \alpha) := \log \inf_{s \in [z^{k\alpha}]} \|s\|_{k\varphi},$$

where  $\|\cdot\|_{k\varphi}$  is the  $L^2$  Hermitian norm induced by  $k\varphi$  on  $H^0(X, kL)$  with respect to a fixed volume form  $\omega^n$ , i.e.

$$\|s\|_{k\varphi}^2 = \int_X |s|^2 e^{-2k\varphi} \omega^n.$$

Witt Nyström then defines the Chebyshev transform  $c[\varphi] : \Delta_{Y_\bullet}(X, L) \rightarrow \mathbb{R}$  to be the convex hull of the functions induced by the  $k^{-1}c_k[\varphi]$  on

$$\pi_{\mathbb{Z}^n}(k^{-1}\Delta_{Y_\bullet}^k(X, L)) \subset \Delta_{Y_\bullet}(X, L).$$

This is a (finite-valued) convex function on the Okounkov body, which generalises the Legendre transform of toric metrics in various ways (see e.g. [15, Theorem 1.4]).

## 2. FLATS IN THE SPACE OF CONTINUOUS PLURISUBHARMONIC METRICS.

We now turn to our main results.

**2.1. Proof of Theorem A, case  $p > 1$ .** The proof of Theorem A will be a direct consequence of the following general result, which itself is a consequence of the main result of [9]. In a nutshell, it states that functions between metric spaces that are not necessarily isometric, but map geodesics to geodesics, preserve flatness via pull-backs.

**Theorem 2.1.** *Let  $(M, d)$ ,  $(M', d')$  be two geodesic metric spaces with  $(M', d')$  isometric to a convex subset of a vector space with strictly convex norm. If there exists a function  $f : M \rightarrow M'$  mapping  $d$ -geodesics to  $d'$ -geodesics, then  $(M, d)$  is also isometric to a convex subset of a vector space with strictly convex norm.*

*Proof.* Let  $x_0, x_1 \in M$ . By [9, Theorem 1.1], since  $(M', d')$  is a convex subset of a vector space with strictly convex norm, there exists a geodesically affine function  $E : M' \rightarrow \mathbb{R}$  which separates  $f(x_0)$  and  $f(x_1)$ , i.e. such that  $E(f(x_0)) \neq E(f(x_1))$ . By our assumption on  $f$ , if  $t \mapsto x_t$  is a  $d$ -geodesic segment in  $M$ , then  $t \mapsto f(x_t)$  is a  $d'$ -geodesic segment in  $M'$ , therefore  $f^*E$  is geodesically affine on  $M$ , and separates  $x_0$  and  $x_1$ . Since one can construct such functions for arbitrary pairs of points in  $M$ , it follows from the other direction of [9, Theorem 1.1] that  $(M, d)$  is also isometric to some convex subset of a strictly convex normed vector space.  $\square$

**Theorem 2.2.** *Let  $\mathcal{U}$  be a  $d_p$ -geodesically convex subset of  $\text{CPSH}(X, L)$ , for any (equivalently, all)  $p > 1$ . If there exists an admissible flag  $Y_\bullet$  of  $X$  such that, for any geodesic segment  $t \mapsto \varphi_t$  lying inside  $\mathcal{U}$ , the Chebyshev transform  $t \mapsto c_{Y_\bullet}[\varphi_t]$  is affine in  $\text{Conv}(\Delta_{Y_\bullet}(X, L))$ , then for all  $p > 1$ ,*

$$(\mathcal{U}, d_p)$$

*is isometric to a convex subset of a vector space endowed with a strictly convex norm.*

*Proof.* This follows on applying Theorem 2.1 with  $(M, d) = (\mathcal{U}, d_p)$ ,  $(M', d') = c_{Y_\bullet}[\mathcal{U}]$  with (e.g.) the norm induced by the  $L^2$ -product on the space of convex functions on  $\Delta_{Y_\bullet}(X, L)$ , and  $f = c_{Y_\bullet}$ .  $\square$

**2.2. Proof of Theorem A, case  $p = 1$ .** For  $p = 1$ , there are in a sense "too many geodesics", and one must use the notion of a *bicombing* introduced in [9], which will allow us to select only Mabuchi geodesics.

**Definition 2.3.** Let  $(M, d)$  be a geodesic metric space. A *bicombing* of  $(M, d)$  is the data, for any two points  $(x_0, x_1) \in M$ , of a unit-parametrised geodesic segment  $t \mapsto x_t^{01}$  connecting  $x_0$  to  $x_1$  such that for all  $t$ ,

$$x_t^{01} = x_{1-t}^{10}.$$

(We note that we only consider unit-parametrised geodesics, while the definition of [9] takes into account arbitrarily parametrised geodesics.)

As explained in [9, Example 1.4], any normed vector space (with non-necessarily strictly convex norm) admits a canonical bicombing, which we will call its *linear bicombing*, sending two given points to the line segment joining them. One can then prove the following version of Theorem 2.1:

**Theorem 2.4.** *Let  $(M, d)$ ,  $(M', d')$  be two geodesic metric spaces with  $(M', d')$  isometric to a convex subset of a Banach space with respect to its linear bicombing. If there exists a bicombing  $\mathcal{B}$  of  $(M, d)$  and a function  $f : M \rightarrow M'$  mapping  $d$ -geodesics in  $\mathcal{B}$  to  $d'$ -geodesics in the linear bicombing of  $(M', d')$ , then  $(M, d)$  is also isometric to a convex subset of a Banach space with respect to its linear bicombing.*

*Proof.* This follows from a proof along the lines of Theorem 2.1 using [9, Corollary 1.4] in stead of [9, Theorem 1.1].  $\square$

In the case where  $M = \text{CPSH}(X, L)$  and  $d = d_p$ ,  $p > 1$ , there exists a unique geodesic between two given points, hence a unique bicombing given by the set of Mabuchi geodesics. For  $p = 1$ , we define  $\mathcal{M}$  to be the bicombing of  $(\text{CPSH}(X, L), d_1)$  given by Mabuchi geodesics. From the previous theorem and the proof of Theorem 2.2 one immediately obtains the following version of Theorem A:

**Theorem 2.5.** *Let  $\mathcal{U}$  be an  $\mathcal{M}$ -geodesically convex subset of  $(\text{CPSH}(X, L), d_1)$ . If there exists an admissible flag  $Y_\bullet$  of  $X$  such that, for any geodesic segment  $t \mapsto \varphi_t$  in  $\mathcal{M} \cap \mathcal{U}$ , the Chebyshev transform  $t \mapsto c_{Y_\bullet}[\varphi_t]$  is affine in  $\text{Conv}(\Delta_{Y_\bullet}(X, L))$ , then  $(\mathcal{U}, d_1)$  is isometric to a convex subset of a Banach space with respect to its linear bicombing.*

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