From norms to metrics in non-Archimedean geometry.

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A plan:

A. An introduction to non-Archimedean geometry.
B. Pluripotential theory in the non-Archimedean world.
C. Finite-energy spaces and geodesics.
D. Interactions with complex pluripotential theory.

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The "real world" is Archimedean. In a non-Archimedean world, this would not necessarily be possible: you could put those segments back to back infinitely many times, and still never reach a certain length.

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To formalize this idea, we first define a **valued field** to be a field $(\mathrm{K}, +, \cdot)$ together with an absolute value, i.e. a function $|\cdot| : \mathrm{K} \to \mathbb{R}_{\geq 0}$ such that:

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This is equivalent to the fact that there exists some real number $\alpha \in \mathbb{R}_{>0}$ such that, for all $n \in \mathbb{N}$,

$$|n1_{\mathrm{K}}| = |1_{\mathrm{K}} + 1_{\mathrm{K}} + \cdots + 1_{\mathrm{K}}| \leq \alpha.$$

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For applications to complex geometry, we will be particularly interested in $(\mathbb{C}, |\cdot|_0)$ and $\mathbb{C}((t))$.

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We want to import these objects to non-Archimedean fields and varieties.

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Problem 1: we cannot naively define manifolds or analytic spaces over a topological space with such properties: we must "add points".

Problem 2: having done that, we will still not be able to perform "differential calculus": we must work *globally* as much as possible.

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Due to the work of Berkovich, we can associate to it an "analytic space" X^{an}, in a functorial manner, and which preserves important topological properties: connectedness, compactness, Hausdorff-ness.

X^{an} is a space made of valuations. In the case where K is $\mathbb{C}$ with the trivial absolute value, X^{an} is simply the set of all valuations on function fields $\mathrm{K}(Y)$ of irreducible subvarieties $Y \subset X$.

We now address our second Problem: having to work globally.

If X is a complex projective manifold, and L is an ample line bundle on X, one can more generally characterize a psh metric on L as a *decreasing limit* of smooth metrics on L with positive curvature form (Demailly '92).

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This characterization is global in nature: we can use it as a definition in the non-Archimedean case, provided we have a "good" class of metrics to take decreasing limits of.

One can also work with **Fubini-Study metrics**, i.e. metrics of the form

$$k^{-1} \log \sum_{i=1}^{h^0(X,kL)} |s_i|^2 e^{2\lambda_i},$$

where $(s_i)_i$ is a basepoint-free basis of the space of sections of kL, and the λ_i are real numbers (or "weights").

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If $L \to X$ are non-Archimedean, one can similarly define a (non-Archimedean) Fubini-Study metric on L^{an}, as a metric of the form

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We then define a (non-Archimedean) **plurisubharmonic metric** on L^{an} to be a decreasing limit (of a net) of Fubini-Study metrics.

Finite-energy spaces and geodesics.

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$$E(\phi_0, \phi_1) = \frac{1}{V(d+1)} \sum_{k=0}^{d} \int_X (\phi_0 - \phi_1)\,(dd^c\phi_0)^k \wedge (dd^c\phi_1)^{d-k},$$

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Because it is decreasing in the first variable, it extends naturally to all of $\mathrm{PSH}(L^{\mathrm{an}})$, possibly taking $-\infty$ as values. We define

$$\mathcal{E}^1(L^{\mathrm{an}}) := \{\phi \in \mathrm{PSH}(L^{\mathrm{an}}),\, E(\phi, \phi_{\mathrm{ref}}) > -\infty \,\forall \phi_{\mathrm{ref}} \in \mathrm{PSH} \cap C^0(L^{\mathrm{an}})\}.$$

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$(\mathcal{E}^1(L^{\mathrm{an}}), d_1)$ *is a geodesic metric space (mod. continuity of envelopes in the non-Archimedean case).*

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In the non-Archimedean case, if the endpoints are continuous, geodesics can furthermore be realized as limits of "Fubini-Study segments"

$$[0,1] \ni t \mapsto k^{-1} \max_i (\log|s_i| - (1-t)\lambda_i - t\lambda_i'),$$

mirroring a result due to Berndtsson in the complex case. To prove this, we "quantize" the non-Archimedean distance via finite-dimensional distances in spaces of norms on each $H^0(X, kL)$.

More on geodesics.

In the complex setting, those geodesics play an important role for studying solutions of the constant scalar curvature Kähler (cscK) equation. Namely, there exists an Euler-Lagrange functional for this equation, the Mabuchi K-energy, which is convex along them.

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They are also detected by the Monge-Ampère energy E, in the sense that a psh segment $t \mapsto \phi_t$ is geodesic if and only if $t \mapsto E(\phi_t, \phi_{\mathrm{ref}})$ is affine for any (hence all) reference metric(s) ϕ_{ref}.

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Theorem (R.)

A non-Archimedean psh segment $t \mapsto \phi_t \in \mathcal{E}^1(L^{\mathrm{an}})$ is geodesic if and only if $t \mapsto E(\phi_t)$ is affine.

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In that sense, n-A psh metrics on L^{an} can be understood as generalized test configurations for (X, L).

Interactions with complex pluripotential theory.

Conversely, there is a natural way to realize n-A psh metrics on L^{an} as complex geometric objects: as *maximal* geodesic rays in $\mathcal{E}^1(L)$.
(Berman-Boucksom-Jonsson)

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Given ϕ^{NA} rational Fubini-Study on L^{an}, i.e. given by a test configuration $(\mathcal{X}, \mathcal{L})$, its maximal geodesic ray is the largest psh ray $t \mapsto \phi_t$ such that its associated metric Φ on $L \times \mathbb{D}^*$ extends as a *locally bounded* psh metric on $\mathcal{L}$.

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More generally, one constructs the maximal ray associated to $\phi^{\mathrm{NA}} \in \mathcal{E}^1(L^{\mathrm{an}})$ as the "least singular" ray Φ with singularity data given by ϕ^{NA}.

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More generally, one constructs the maximal ray associated to $\phi^{\mathrm{NA}} \in \mathcal{E}^1(L^{\mathrm{an}})$ as the "least singular" ray Φ with singularity data given by ϕ^{NA}. Such rays have played an essential role in the variational proofs of various versions of the Yau-Tian-Donaldson conjecture (BBJ, Li, Han-Li...)

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Denote by $\mathcal{R}^1(L)$ the space of finite-energy psh rays (emanating from ϕ_0). It can be metrized via the "slope at infinity"

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Theorem (Berman-Boucksom-Jonsson '15)

The mapping sending ϕ^{NA} to its associated maximal ray is an isometric (i.e. distance-preserving and injective) map from $(\mathcal{E}^1(L^{\mathrm{an}}), d_1^{\mathrm{NA}})$ to $(\mathcal{R}^1(L), \hat{d}_1)$.

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We consider a holomorphic submersion $\pi : (X, L) \to \mathbb{D}^*$ over the punctured unit disc, with L π-ample, such that there exists an analytic model for (X, L).

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On the one hand, we can realize X and L as varieties X_{K} and L_{K} over $\mathrm{K} = \mathbb{C}((t))$. We thus naturally have a space of n-A finite-energy metrics $(\mathcal{E}^1(L_{\mathrm{K}}^{\mathrm{an}}), d_1^{\mathrm{NA}})$.

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One can similarly as in $\mathcal{R}^1$ define:

1. the "maximal family" in $\mathcal{D}^1(L)$ corresponding to a given metric $\phi^{\mathrm{NA}} \in \mathcal{E}^1(L_{\mathrm{K}}^{\mathrm{an}})$, which is defined as a solution to an envelope problem again, and correspond to maximal rays in the invariant case;
2. a distance $\hat{d}_1$ on this space via Lelong numbers $\nu_0(z \mapsto d_1(\phi_{0,z}, \phi_{1,z}))$.

(for the last point, we require additional technical hypotheses mimicking geodesicity.)

Theorem (R. '21)

The map sending ϕ^{NA} to its maximal family is an isometric map from $(\mathcal{E}^1(L_{\mathrm{K}}^{\mathrm{an}}), d_1^{\mathrm{NA}})$ to $(\mathcal{D}^1(L), \hat{d}_1)$. In particular, both spaces are complete, geodesic metric spaces.

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2. Interpreting non-Archimedean K-stability over $\mathbb{C}((t))$?
3. Convexity of some important functionals (K^{NA}, H^{NA}) and geometric properties of their minimizers.

Thank you for your attention !